Coordinated logistics with a truck and a drone

Abstract

We determine the efficiency of a delivery system in which an unmanned aerial vehicle (UAV) provides service to customers while making return trips to a truck that is itself moving. In other words, a UAV picks up a package from the truck (which continues on its route), and after delivering the package, the UAV returns to the truck to pick up the next package. Although the hardware for such systems already exists, it is not yet understood to what extent such an approach can actually provide a significantly improved quality of service. By combining a theoretical analysis in the Euclidean plane with real-time numerical simulations on a road network, we demonstrate that the improvement in efficiency is related to the square root of the ratio of the speeds of the truck and the UAV.

1 Introduction

One of the most talked-about developments in transportation and logistics in recent years has been the potential use of unmanned aerial vehicles (UAVs), or “drones”, for transporting packages, food, medicine, and other goods. The most famous proof-of-concept of such a service is the “Amazon Prime Air” system, which was introduced in late 2013 and has since undergone several iterations [10]. Other similar systems include Google’s “Project Wing” [37], DHL’s “Parcelcopter” [12], and a joint effort between the Swiss Post, Swiss Worldcargo (the air freight division of Swiss International Air Lines), and the California-based startup Matternet [19]. As described in [33], there are many reasons to be optimistic about the role that UAVs can play in next-generation logistics systems:
[Demand for last-mile delivery] is likely to increase as e-commerce volumes grow. ... UAVs could provide major relief for inner cities, taking traffic off the roads and into the skies. So far, payloads are limited but a network of UAVs could nevertheless support first and last-mile logistics networks. ... This urban first and last-mile use case is probably the most tangible and spectacular in the logistics industry.

UAVs are also already being used in many related industries including energy [48], agriculture and forestry [4], environmental protection [42], and emergency response [41].

From a transportation scientist’s perspective, many of the benefits of a UAV-based delivery system are obvious: UAVs have a low per-mile cost, can operate without human intervention, and can travel at high speeds while being unaffected by road traffic. The shortcomings of such a system are equally apparent: UAVs have an extremely low carrying capacity and short travelling radius, both of which necessitate frequent returns to a central depot. Thus, as suggested in Figures 1a and 1b, a conventional truck delivery system benefits from an economy of scale and suffers from high per-mile costs, whereas a UAV delivery system benefits from low per-mile costs and lacks an economy of scale.

The purpose of this paper is to determine the efficiency of a hybrid approach in which a UAV provides service to customers while making return trips to a truck that is itself moving, as illustrated in Figure 1c. In other words, a UAV picks up a package from the truck (which continues on its route), and after delivering the package, the UAV returns to the truck to pick up the next package. Although the hardware for such systems has already been constructed – one particular implementation is called the “HorseFly” and was developed jointly with AMP Electric Vehicles and the University of Cincinnati [54] – it is not yet understood to what extent such an approach can actually provide a significantly improved quality of service. Our goal is to describe precisely how much improvement can be realized in a mathematically sound way.

Obviously, the “horsefly” problem is extremely difficult to solve to optimality because it is
Figure 1: (a) shows a travelling salesman tour of a set of client destinations and a central depot, which corresponds to the optimal provision of service by a delivery scheme that uses only a truck. Similarly, (b) shows a collection of direct trips between the same set of points and the central depot, which corresponds to the optimal provision of service by a delivery scheme that uses only UAVs. Finally, (c) shows the solution to a problem in which we have a truck that follows a route that is much shorter than that of (a), as indicated by the thick tour, as well as a single UAV that alternates between visiting the client destinations and and the truck, as indicated by the dashed lines. The routes shown correspond to the case where the UAV is about four times as fast as the truck; it is also worth mentioning that the optimal sequence of the client destinations is not necessarily the same as that induced by the TSP tour which is shown in (a), and indeed one can readily observe that there are several places where the two sequences differ.

a generalization of the travelling salesman problem (TSP) that requires the consideration of the locations where the truck and the UAV meet. Thus, we approach this problem from the continuous approximation paradigm, in which “detailed data are replaced by concise summaries, and numerical methods are replaced by analytic models” [20]; our goal is to reduce the problem to a small set of parameters, and then determine how these parameters affect the outcome of the problem. In a nutshell, our model assumes that customers are distributed according to a known probability density in the Euclidean plane. We primarily focus on the time to completion of all of the deliveries, and our results are also easily applicable to other cost functions such as the net energy expenditure over both vehicles.

The remainder of this paper is structured as follows: Section 2 defines our problem formally and also gives a pair of geometric results. Section 3 gives upper and lower bounds for the time to completion of a tour under the assumption that demand is continuously distributed in the Euclidean
plane, and characterizes the amount of improvement that is realized by a hybrid system in terms of the speeds of the truck and the UAV. Finally, Section 4 describes two computational simulations that verify that our continuous approximation result is a valid one in practice; one simulation is conducted in the Euclidean plane, whereas the other uses real-time driving information on a road network using the Google Directions API [3].

1.1 Related work

The use of UAVs in logistics is a prospect that is very much in its infancy, and as such, there is little research on the economics of such systems. The recent papers [5, 30, 44] all describe mixed integer programming formulations and heuristic algorithms for solving several optimization problems in which a truck and a UAV perform service in tandem. Even more recently, [53] describes “worst-case” configurations of input points that minimize the benefit that a UAV can offer in assisting a truck; their analysis further enables one to make “best-case” predictions as well, by establishing that the improvement offered by adding a UAV (or multiple UAVs) cannot exceed a certain value. Instead of examining solutions to specific problem instances or examining worst- or best-case scenarios, our paper – by comparison – is concerned with the long-term behavior when many customers are serviced in a region according to a population density (this is the principle of the continuous approximation paradigm for transportation, which we discuss later in this section). The most concrete discussion of the economic desirability thus far can be found in [22], which analyzes per-mile costs based on the estimated energy consumption of UAVs and describes several feasibility issues in implementation; the author concludes that “From a cost perspective, the numbers do not look unreasonable”. Within the operations research community, extensive research has been conducted on the use of UAVs for military applications; for example, [24] uses robust optimization to plan reconnaissance missions subject to uncertainty in fuel usage between locations and weather conditions and [39] uses scenario-based optimization to decide the number of UAVs to deploy; once
deployed, more information about the environment is learned, at which point the route must be
determined.

Our problem can essentially be thought of as an intermodal instance of the Vehicle Routing
Problem (VRP) [26, 51]. Because we seek to optimally coordinate two classes of vehicles that have
diametrically opposing strengths and weaknesses, our problem is particularly related to instances
of VRP in which vehicle heterogeneity plays an important role, such as [27]. A closely related
problem to ours is described in [40], which gives an integer programming formulation and a heuristic
algorithm for solving a routing problem in which a truck can carry a fleet of “foot couriers” on a
single- or multi-route assignment, and the goal is to coordinate resources between the truck and
the couriers effectively. The problem of coordinating two separate classes of vehicles is most widely
encountered in the truck and trailer problem in the VRP community [18, 47, 52], in which a vehicle
carries a large trailer that must be employed at certain destinations.

One of the basic phenomena that is of interest to us is the trade-off between efficiency in trans-
portation along a backbone network (in this case, the route of the truck) versus direct trips between
locations (in this case, the direct trips taken by UAVs); this is arguably one of the fundamental
dichotomies in transportation and logistics [15, 16]. In this sense, our problem of interest is philo-
sopherically similar is [14], which asks whether small local retail stores are preferable to “big-box”
retailers, with [55], which estimates the changes in net CO\textsubscript{2} emissions that result by introducing
grocery delivery services, and with [56], which computes the optimal layout of a set of facility
locations that are themselves connected with a backbone network.

This paper is concerned with a continuous approximation model for a transportation problem,
and is therefore philosophically similar to (for example) [13], which analytically determines trade-
offs between transportation and inventory costs, [35], which shows how to route emergency relief
vehicles to beneficiaries in a time-sensitive manner, and [36], which describes a simple geometric
model for determining the optimal mixture of a fleet of vehicles that perform distribution. The basic
premise of the continuous approximation paradigm is that one replaces combinatorial quantities that are difficult to compute with simpler mathematical formulas, which (under certain conditions) provide accurate estimations of the desired quantity (and indeed, we will verify in Section 4 that our theoretical analysis holds under realistic modelling assumptions). Such approximations exist for many combinatorial problems, such as the travelling salesman problem [9, 25], facility location [31, 34, 45], and any subadditive Euclidean functional such as a minimum spanning tree, Steiner tree, or matching [46, 49, 50]. Our particular usage of the continuous approximation paradigm is that we treat the customer demand as coming from a probability density function; in this sense our work also shares some commonality with stochastic vehicle routing problems [17, 29].

1.2 Notational conventions

Our notational conventions are as follows: we assume that there are $n$ customer locations in the Euclidean plane to be visited with a truck and a UAV whose speeds are $\phi_0$ and $\phi_1$ respectively, with $\phi_0 < \phi_1$. These customers are assumed to follow an absolutely continuous probability distribution $f$, which is defined on a compact planar region $\mathcal{R}$. We use $\mathcal{L}$ to denote a loop in $\mathcal{R}$ (representing the truck’s tour), we use $\text{Loop}(\mathcal{R})$ to denote the set of all loops $\mathcal{L}$ in $\mathcal{R}$ whose length is well-defined, and we let $d(x, \mathcal{L})$ denote the distance between point $x$ to loop $\mathcal{L}$; that is,

$$d(x, \mathcal{L}) = \min_{x' \in \mathcal{L}} \|x - x'\|,$$

where $\|\cdot\|$ is the usual Euclidean distance. The set of permutations of $\{1, \ldots, n\}$ will be written as $S_n$, with a particular permutation written as $\sigma \in S_n$; because these permutations always correspond to a tour that our truck takes, we will adopt the convention that $\sigma(n + 1) = \sigma(1)$ for brevity. The (closed) $\epsilon$-neighborhood of a (compact) set $\mathcal{S}$ in the plane will be written as $N_\epsilon(\mathcal{S})$, which is to say,

$$N_\epsilon(\mathcal{S}) = \{x \in \mathbb{R}^2 : \min_{x' \in \mathcal{S}} \|x - x'\| \leq \epsilon\}.$$
We say that a function $g(x)$ satisfies $g(x) \sim h(x)$ as $x \to \infty$ if $\lim_{x \to \infty} g(x)/h(x) = 1$. Finally, we use the expression $\mathbbm{1}_{x \in S}$ to denote the indicator function for membership in set $S$.

2 Preliminaries

We begin by formally defining the *horsefly routing problem* in which we coordinate a truck and a UAV:

**Definition 1** (Horsefly routing). Let $p_1, \ldots, p_n$ be a collection of points in the plane and let $\phi_0, \phi_1 > 0$ denote the speeds of a truck and a UAV respectively, with $\phi_0 < \phi_1$. The optimal *horsefly tour* of $p_1, \ldots, p_n$ is the solution to the optimization problem

$$\min_{x_1, \ldots, x_n, \sigma \in S_n} \sum_{i=1}^{n} \max \left\{ \frac{1}{\phi_0} \|x_{\sigma(i)} - x_{\sigma(i+1)}\|, \frac{1}{\phi_1} \left( \|x_{\sigma(i)} - p_{\sigma(i)}\| + \|p_{\sigma(i)} - x_{\sigma(i+1)}\| \right) \right\}$$

(1)

where $S_n \ni \sigma$ is the set of all permutations of the set $\{1, \ldots, n\}$, with the added convention that $\sigma(n+1) = \sigma(1)$ for notational convenience.

In the above problem, each variable $x_i$ corresponds to the “launch site” at which the truck releases the UAV to visit customer point $p_i$; the first term in the $\max\{\cdot, \cdot\}$ expression simply corresponds to the amount of time needed for the truck to move from one launch site to the next, whereas the second term corresponds to the amount of time needed for the UAV to leave its launch site, arrive at its customer point, and return to rendezvous with the truck at the next launch site.

It is worth noting that, if one fixes the permutation $\sigma$, the remaining optimization problem over variables $x_i$ is convex (a second order cone program). Also note that (1) is a generalization of the Euclidean TSP (which is the limiting case $\phi_1 \to \phi_0$), and hence its objective value is always at most equal to the cost of a TSP tour of the points $p_i$ with the truck; this corresponds to setting $x_i = p_i$ for all $i$.

The following classical theorem, originally stated in [9] and further developed in [49, 50], relates
the length of a TSP tour of a sequence of points with the distribution from which they were sampled:

**Theorem 2** (BHH Theorem). Suppose that $X = \{X_1, X_2, \ldots \}$ is a sequence of random points i.i.d. according to an absolutely continuous probability density function $f$ defined on a compact planar region $\mathcal{R}$. Then with probability one, the length $TSP(X)$ of the optimal travelling salesman tour through $X$ satisfies

$$
\lim_{N \to \infty} \frac{TSP(X)}{\sqrt{N}} = \beta \int_{\mathcal{R}} \sqrt{f(x)} \, dx
$$

where $\beta$ is a constant.

Although the exact value of $\beta$ is unknown, it has been shown that $0.6250 \leq \beta \leq 0.9204$; see [6, 9].

The following geometric result will be helpful to us later in relating the workload of the truck and the workload of the UAV; given a loop $\mathcal{L}$ (which represents the tour taken by the truck), it is useful to know how much area is within a given distance $\epsilon$ of $\mathcal{L}$:

**Lemma 3.** Let $\mathcal{L}$ be a loop in the plane with length $\ell$ and let $\epsilon > 0$. The area of an $\epsilon$-neighborhood of $\mathcal{L}$ is at most

$$
\text{Area}(N_\epsilon(\mathcal{L})) \leq \begin{cases} 
2\epsilon\ell & \text{if } \epsilon \leq \frac{\ell}{2\pi} \\
\pi\epsilon^2 + \epsilon\ell + \frac{\ell^2}{4\pi} & \text{otherwise,}
\end{cases}
$$

which is tight when $\mathcal{L}$ is a circle, whereby $N_\epsilon(\mathcal{L})$ is either an annulus (if $\epsilon \leq \ell/(2\pi)$) or a disk (if $\epsilon > \ell/(2\pi)$).

**Proof.** Assume without loss of generality that $\mathcal{L}$ forms the boundary of a convex region and that $\mathcal{L}$ is piecewise linear, i.e. polygonal (see Section A of the online supplement for justifications of both of these assumptions). The $\epsilon$-neighborhood of $\mathcal{L}$ has an “inner” portion $R_{in}$ and an “outer” portion $R_{out}$ as shown in Figure 2a, and we can also see from Figure 2a that the area of the “outer” portion $R_{out}$ is always exactly $\pi\epsilon^2 + \epsilon\ell$. It is also obvious that the outer perimeter of $R_{out}$ is exactly $\ell + 2\pi\epsilon$. The area of the “inner” portion $R_{in}$ is a little more complicated to bound; first, for any $\epsilon' \leq \epsilon$,
we let \( L_{\epsilon'} \) denote the closed curve inside \( L \) consisting of points that are exactly \( \epsilon' \) away from their nearest point in \( L \) (thus, our original \( L \) would simply be written as \( L_0 \) under this notation). It is of course possible that \( L_{\epsilon'} = \emptyset \) for sufficiently large \( \epsilon' \). Since \( R_{in} \) is simply the union of all curves \( L_{\epsilon'} \) over \( \epsilon' \in [0, \epsilon] \), the coarea formula [38] says that the area of \( R_{in} \) is obtained by integrating the length of \( L_{\epsilon'} \) from \( \epsilon' = 0 \) to \( \epsilon' = \epsilon \):

\[
\text{Area}(R_{in}) = \int_0^\epsilon \text{length}(L_{\epsilon'}) \, d\epsilon'.
\]

Let \( R_{out}' \) denote the “outer” portion of the \( \epsilon' \)-neighborhood of \( L_{\epsilon'} \), as shown in Figure 2b. We then have \( R_{out}' \subseteq R_{in} \), which by convexity implies that the outer perimeter of \( R_{out}' \) is less than or equal to the length of \( L \), i.e. \( \ell \). However, by the same reasoning as our calculation of the outer perimeter of \( R_{out} \), we also see that the outer perimeter of \( R_{out}' \) is exactly \( \text{length}(L_{\epsilon'}) + 2\pi \epsilon' \), and therefore

\[
\ell = \text{length}(L) \geq \text{perimeter}(R_{out}') = \text{length}(L_{\epsilon'}) + 2\pi \epsilon'
\]

\[
\implies \text{length}(L_{\epsilon'}) \leq \ell - 2\pi \epsilon'
\]

provided that \( L_{\epsilon'} \) exists, i.e. that \( \text{length}(L_{\epsilon'}) > 0 \). Thus, we see that

\[
\text{Area}(R_{in}) = \int_0^\epsilon \text{length}(L_{\epsilon'}) \, d\epsilon' \leq \int_0^\epsilon \max\{\ell - 2\pi \epsilon', 0\} \, d\epsilon' = \begin{cases} 
\ell \epsilon - \pi \epsilon^2 & \text{if } \epsilon \leq \frac{\ell}{2\pi} \\
\frac{\epsilon^2}{4\pi} & \text{otherwise}
\end{cases}
\]

from which the desired result follows, since \( \text{Area}(N_\epsilon(L)) = \text{Area}(R_{in}) + \text{Area}(R_{out}) \).
Figure 2: (a) shows the neighborhoods $R_{in}$ and $R_{out}$; note that $R_{out}$ is the union of the hatched circular sectors and shaded rectangles, and therefore $\text{Area}(R_{out}) = \pi \epsilon^2 + \epsilon \ell$ and $\text{perimeter}(R_{out}) = \ell + 2\pi \epsilon$. (b) shows an inner loop $L_{\epsilon'}$ and illustrates that $R'_{out} \subseteq R_{in}$ for all $\epsilon'$ (which characterize $R'_{out}$).

3 Continuous approximation analysis of the horsefly routing problem

This section derives upper and lower bounds for a continuous approximation of problem (1), under the assumption that the customers are distributed according to a known absolutely continuous probability distribution $f$. Not surprisingly, it is useful to describe the average distance between a customer sampled from $f$ and a loop $L$:

**Theorem 4.** Let $\mathcal{R}$ be a compact planar region and let $f$ be an absolutely continuous probability density function defined on $\mathcal{R}$. Let $\text{OPT}(\ell)$ denote the optimal objective value to the problem

$$\minimize_{L \in \text{Loop}(\mathcal{R})} \iint_{\mathcal{R}} f(x) d(x, L) \, dx \quad s.t. \quad \text{length}(L) = \ell,$$

where the optimization variable $L$ is taken over the set of all loops in $\mathcal{R}$ whose length is well-defined.
Then

\[
\text{OPT}(\ell) \sim \frac{1}{4\ell} \left( \int_{\mathcal{R}} \sqrt{f(x)} \, dx \right)^2
\]

as \( \ell \to \infty \).

Figure 3d shows an example of an optimal solution to this problem. In order to prove Theorem 4, it is easiest first to consider the special case where \( f \) is a uniform distribution:

Lemma 5. Let \( \mathcal{L} \) be a loop in a compact region \( \mathcal{R} \) with area \( A \). Then \( \int_{\mathcal{R}} d(x, \mathcal{L}) \, dx \) satisfies

\[
\int_{\mathcal{R}} d(x, \mathcal{L}) \, dx \geq \begin{cases} 
\frac{A^{3/2}}{3\sqrt{\pi}} - \frac{A\ell}{2\pi} + \frac{\ell^3}{12\pi^2} & \text{if } \ell \leq \sqrt{A\pi} \\
\frac{A^2}{4\ell} & \text{otherwise.}
\end{cases}
\]

(3)

Proof. We can assume without loss of generality that \( A = 1 \) because, if we apply the transformation \( A \mapsto cA \) and \( \ell \mapsto \sqrt{c\ell} \) for any \( c > 0 \), then the right-hand side of the above is scaled by \( c^{3/2} \). Thus, the quantity of interest \( \int_{\mathcal{R}} d(x, \mathcal{L}) \, dx \) is simply equal to the expected distance between a point \( X \) uniformly sampled in \( \mathcal{R} \) and \( \mathcal{L} \), i.e. \( \mathbb{E}d(X, \mathcal{L}) \). Recall that for any non-negative random variable \( Z \) on the real line, we have

\[
\mathbb{E}(Z) = \int_{0}^{\infty} 1 - F(z) \, dz ,
\]

where \( F \) is the cumulative distribution function of \( Z \) (this is a simple consequence of integration by parts \([32]\)). If we set \( Z = d(X, \mathcal{L}) \), where \( X \) is uniformly sampled in \( \mathcal{L} \), then Lemma 3 tells us that

\[
F(z) = \text{Area}(N_z(\mathcal{L}) \cap \mathcal{R}) \leq \text{Area}(N_z(\mathcal{L})) \leq \begin{cases} 
2z\ell & \text{if } z \leq \frac{\ell}{2\pi} \\
\pi z^2 + z\ell + \frac{\ell^2}{4\pi} & \text{otherwise.}
\end{cases}
\]

Suppose that \( \ell > \sqrt{\pi} \) as in the second case of the desired inequality. We then have

\[
\int_{\mathcal{R}} d(x, \mathcal{L}) \, dx = \mathbb{E}d(X, \mathcal{L}) = \int_{0}^{\infty} 1 - F(z) \, dz \geq \int_{0}^{\frac{\ell}{2\pi}} 1 - F(z) \, dz \geq \int_{0}^{\frac{\ell}{2\pi}} 1 - 2z\ell \, dz = \frac{1}{4\ell} .
\]
Figure 3: Figures (a) and (b) show “zig-zagging” and “spiralling” tours through a region $\mathcal{R}$ with a loop $\mathcal{L}$, in such a way that $\iint_{\mathcal{R}} d(x, \mathcal{L}) \, dx \sim A^2/(4\ell)$ as $\ell \to \infty$. When the demand density $f$ is non-uniform, an optimal loop should be more “concentrated” in denser areas, as shown in (c) and (d); the precise nature of this concentration, as well as the asymptotic costs incurred, are described in the proof of Theorem 4. In the interest of full disclosure, we remark that the loop in (d) was sketched by hand and then locally improved; it is not guaranteed to be optimal, but merely suggest the overall contour that one will see as $\ell \to \infty$.

On the other hand, if $\ell \leq \sqrt{\pi}$ as in the first case of the desired inequality, we have $\ell/(2\pi) \leq 1/\sqrt{\pi} - \ell/(2\pi)$, whence

$$\iint_{\mathcal{R}} d(x, \mathcal{L}) \, dx = \int_0^\infty 1 - F(z) \, dz \geq \int_0^{\ell/2\pi} 1 - F(z) \, dz + \int_{\ell/2\pi}^{\ell/\sqrt{\pi}} 1 - F(z) \, dz$$

$$\geq \int_0^{\ell/2\pi} 1 - 2z\ell \, dz + \int_{\ell/2\pi}^{\ell/\sqrt{\pi}} 1 - (\pi z^2 + z\ell + \ell^2/4\pi) \, dz = \frac{2}{3\sqrt{\pi}} - \frac{\ell}{2\pi} + \frac{\ell^3}{12\pi^2}$$

which completes the proof. \qed

Remark 6. As in Lemma 3, the lower bound (3) is tight when $\mathcal{L}$ is a circle and $\mathcal{R}$ is either an annulus or a disk. When $\mathcal{R}$ is an arbitrary region, we can construct a family of loops $\mathcal{L}$ with length $\ell$ that satisfy $\iint_{\mathcal{R}} d(x, \mathcal{L}) \, dx \sim A^2/(4\ell)$ as $\ell \to \infty$ by “zig-zagging” or “spiralling” through $\mathcal{R}$, as shown in Figures 3a and 3b; see Section B of the Online Supplement for further details, as well as [8, 11] for computational geometric analysis. In the non-uniform case, the optimal solution to (2) is to zig-zag or spiral in a way that is consistent with the demand density $f$, as can be seen in Figure 3d.

Theorem 4 is now straightforward:
Proof of Theorem 4. It will suffice to show that, given any threshold $\epsilon > 0$ together with $R$ and $f$, it is always possible to select a length $\bar{\ell}$ such that the optimal objective value to problem (2), i.e. $\text{OPT}(\ell)$, satisfies

$$1 - \frac{\epsilon}{4\bar{\ell}} \left( \int_R \sqrt{f(x)} \, dx \right)^2 \leq \text{OPT}(\ell) \leq 1 + \frac{\epsilon}{4\bar{\ell}} \left( \int_R \sqrt{f(x)} \, dx \right)^2$$

for all $\ell \geq \bar{\ell}$. Because $f$ is absolutely continuous, it is possible to approximate $f$ arbitrarily well with a step function $\tilde{f} = \sum_{i=1}^{N} \tilde{f}_i$, where each component $\tilde{f}_i$ is a constant function on a patch $P_i \subset R$. That is, $\tilde{f}_i(x) = a_i \mathbf{1}_{x \in P_i}$ for positive scalars $a_i$, where $\mathbf{1}_{x \in P_i}$ is the indicator function for membership in $P_i$. Given any loop $L$ in $R$, it is always possible to create a collection of loops $L_i \subset P_i$ such that $L \subseteq \bigcup_i L_i$ and $\sum_i \text{length}(L_i) \leq \text{length}(L) + c_1$, where $c_1$ is a constant that depends only on the patches $P_i$ (one acceptable value of $c_1$ is the sum of the perimeters of the $P_i$’s); this is illustrated in Figures 4a and 4b. Similarly, given any collection of loops $L_i \subset P_i$, it is always possible to create a single loop $L$ such that $\bigcup_i L_i \subseteq L$ and such that $\text{length}(L) \leq \sum_i \text{length}(L_i) + c_2$, where $c_2$ again depends only on the patches (and again, an acceptable value of $c_2$ is the sum of the perimeters of the $P_i$’s); this is illustrated in Figures 4c and 4d. These observations are useful to us because we are interested in the limiting behavior of (2) as $\ell \to \infty$, and we see that $c_1$ and $c_2$ are independent of $\ell$.

Given a desired threshold $\epsilon > 0$, choose $\delta > 0$ so that $(1 - \epsilon) \leq (1 - \delta)^2 < (1 + \delta)^2 \leq (1 + \epsilon)$ and let $\tilde{f}$ be a sufficiently fine approximation that $(1 - \delta)\tilde{f}(x) \leq f(x) \leq (1 + \delta)\tilde{f}(x)$ for all $x \in R$. For reasons that will be clear later, we will also require that $\tilde{f}$ be chosen so that

$$1 - \delta \leq \frac{\int_R \sqrt{f(x)} \, dx}{\int_R \sqrt{\tilde{f}(x)} \, dx} \leq 1 + \delta.$$
Figure 4: In the above diagrams, we have $N = 16$ square patches $\mathcal{P}_i$. Figures (a) and (b) show that one can always decompose a single loop into multiple loops in a way that depends only on the shapes of the patches. Figures (c) and (d) show that one can always join a collection of loops into a single loop in a way that also depends only on the shapes of the patches.

By our previous observations, $\text{OPT}(\ell)$ is bounded above by the optimal solution to

$$\min_{\mathcal{L}_1, \ldots, \mathcal{L}_N} (1 + \delta) \sum_{i=1}^{N} \iint_{\mathcal{P}_i} \tilde{f}_i(x) d(x, \mathcal{L}_i) \, dx$$

$$\sum_{i=1}^{N} \text{length}(\mathcal{L}_i) = \ell - c_2,$$

where each $\mathcal{L}_i$ is a loop in patch $\mathcal{P}_i$. Consider the problem of selecting a single optimal loop $\mathcal{L}_i \subset \mathcal{P}_i$, under the assumption that length($\mathcal{L}_i$) = $\ell_i$ is known; this is written as

$$\min_{\mathcal{L}_i \in \text{Loop}(\mathcal{P}_i)} \iint_{\mathcal{P}_i} f_i(x) d(x, \mathcal{L}_i) \, dx \quad \text{s.t.} \quad \text{length}(\mathcal{L}_i) = \ell_i,$$

and since $\tilde{f}_i(x) = a_i$ on $\mathcal{P}_i$, we therefore can apply Lemma 5 and Remark 6, which say that, for sufficiently large $\ell_i$, the optimal objective value of (5) is asymptotically equal to $a_i \text{Area}(\mathcal{P}_i)^2/(4\ell_i)$; moreover, this is realizable by “zig-zagging” or “spiralling” in $\mathcal{P}_i$ as in Figure 3. Thus, we see that
as $\ell \to \infty$, problem (4) is asymptotically equivalent to the problem

$$\begin{align*}
\text{minimize} \quad & (1 + \delta) \sum_{i=1}^{N} a_i \cdot \frac{\text{Area}(P_i)^2}{4\ell_i} \\
\text{s.t.} \quad & \sum_{i=1}^{N} \ell_i = \ell - c_2 \\
& \ell_i \geq 0
\end{align*}$$

as $\ell \to \infty$. The optimal solution to (6) is to set

$$\ell_i^* = \frac{\sqrt{a_i \text{Area}(P_i)}}{\sum_{i=1}^{N} \sqrt{a_i \text{Area}(P_i)}} \cdot (\ell - c_2)$$

for each $i$, which gives an objective value of

$$(1 + \delta) \sum_{i=1}^{N} a_i \cdot \frac{\text{Area}(P_i)^2}{4\ell_i^*} = \frac{1 + \delta}{4(\ell - c_2)} \left( \sum_{j=1}^{N} \sqrt{a_i \text{Area}(P_i)} \right)^2 = \frac{1 + \delta}{4(\ell - c_2)} \left( \int_{\mathcal{R}} \sqrt{\bar{f}(x)} \, dx \right)^2$$

$$\leq \frac{(1 + \delta)^2}{4(\ell - c_2)} \left( \int_{\mathcal{R}} \sqrt{f(x)} \, dx \right)^2$$

$$\implies \text{OPT}(\ell) \leq \frac{1 + \epsilon}{4(\ell - c_2)} \left( \int_{\mathcal{R}} \sqrt{f(x)} \, dx \right)^2.$$  

We can also derive a lower bound of (2) using nearly identical reasoning, because $\text{OPT}(\ell)$ is bounded below by the optimal solution to

$$\begin{align*}
\text{minimize} \quad & (1 - \delta) \sum_{i=1}^{N} \int_{\mathcal{P}_i} \tilde{f}_i(x) d(x, \mathcal{L}_i) \, dx \\
\text{s.t.} \quad & \sum_{i=1}^{N} \text{length}(\mathcal{L}_i) = \ell + c_1
\end{align*}$$

and an entirely analogous argument shows that

$$\text{OPT}(\ell) \geq \frac{1 - \epsilon}{4(\ell + c_1)} \left( \int_{\mathcal{R}} \sqrt{f(x)} \, dx \right)^2$$

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for sufficiently large $\ell$, which completes the proof.

3.1 A lower bound

To bound problem (1) from below, the following geometric lemma is useful:

**Lemma 7.** Let $q_1, q_2, q_3$ be any three points in $\mathbb{R}^2$, and let $s$ denote the distance from $q_3$ to the line segment $q_1q_2$. We have

$$\|q_1 - q_3\| + \|q_2 - q_3\| \geq \sqrt{\|q_1 - q_2\|^2 + 4s^2}. \quad (7)$$

**Proof.** Assume without loss of generality that $q_1 = (-1, 0)$, $q_2 = (1, 0)$ and that $q_3 = (u, v)$, with $u, v \geq 0$. Further assume that $u \leq 1$, which implies that $s = v$; we address the case $u > 1$ in the next paragraph. We have $s = v$ and $\|q_1 - q_2\| = 2$, so that the right-hand side of (7) is independent of $u$.

By convexity, we see that the choice of $u$ that minimizes the left-hand side of (7) is to set $u = 0$, so that $q_3$ lies on the perpendicular bisector of $q_1q_2$. Thus, we have $\|q_1 - q_3\| = \|q_2 - q_3\| = \sqrt{1 + v^2}$ and therefore

$$\sqrt{1 + v^2} + \sqrt{1 + v^2} = \sqrt{4 + 4v^2} = \sqrt{2^2 + 4s^2} = \sqrt{\|q_1 - q_2\|^2 + 4s^2}$$

as desired (the inequality of (7) is actually an equality when $q_3$ lies on the perpendicular bisector of $q_1q_2$).

To address the case where $u > 1$, we let $r_1 = \|q_1 - q_3\|$ and $r_2 = \|q_2 - q_3\|$, and we let $q'_3 = (1, r_2)$; note that the right-hand side of (7) is the same at both $q_3$ and $q'_3$ (and $q'_3$ satisfies (7) by the preceding paragraph). Thus, it will suffice to verify that the left-hand side of (7) does not increase if we replace $q_3$ with $q'_3$. This is simple: let $C_1$ denote the circle of radius $r_1$ about $q_1$ (which has $q_3$ on its boundary by construction) and let $C_2$ denote the circle of radius $r_2$ about $q_2$ (which has both $q_3$ and $q'_3$ on its boundary); since two distinct circles intersect in at most two
points, and since $C_1$ and $C_2$ both contain $q_3$ and its vertical reflection $\bar{q}_3 = (u, v)$, it follows that the counterclockwise arc on $C_2$ from $q_3$ to $\bar{q}_3$ must be in the interior of $C_1$. Hence, $q'_3$ is in the interior of $C_1$, so that $\|q_1 - q'_3\| < r_1$. Since $\|q_2 - q'_3\| = r_2 = \|q_2 - q_3\|$, this completes the proof. □

It is fairly straightforward to derive a lower bound of problem (1) that is more amenable to asymptotic analysis:

**Claim 8.** The optimal objective value to the problem

$$\minimize_{L \in \text{Loop}(R)} \max \left\{ \frac{1}{\phi_0} \text{length}(L), \frac{1}{\phi_1} \sqrt{\text{length}(L)^2 + 4 \left( \sum_{i=1}^{n} d(p_i, L) \right)^2} \right\}$$

is a lower bound of problem (1).

**Proof.** Assume without loss of generality (purely for notational convenience) that the optimal permutation $\sigma^*$ to (1) is the identity (with the same convention that that $n + 1$ is equivalent to 1). For each $i$, let $s_i$ denote the distance from $p_i$ to the line segment $x_i x_{i+1}$. Applying Lemma 7 and letting $L$ be the tour that traverses the points $x_i$ in order, we have

$$\sum_{i=1}^{n} \max \left\{ \frac{1}{\phi_0} \|x_i - x_{i+1}\|, \frac{1}{\phi_1} \left( \|x_i - p_i\| + \|p_i - x_{i+1}\| \right) \right\}$$

$$\geq \sum_{i=1}^{n} \max \left\{ \frac{1}{\phi_0} \|x_i - x_{i+1}\|, \frac{1}{\phi_1} \sqrt{\|x_i - x_{i+1}\|^2 + 4s_i^2} \right\}$$

$$\geq \max \left\{ \frac{1}{\phi_0} \sum_{i=1}^{n} \|x_i - x_{i+1}\|, \frac{1}{\phi_1} \sum_{i=1}^{n} \sqrt{\|x_i - x_{i+1}\|^2 + 4s_i^2} \right\}$$

$$\geq \max \left\{ \frac{1}{\phi_0} \text{length}(L), \frac{1}{\phi_1} \sum_{i=1}^{n} \sqrt{\|x_i - x_{i+1}\|^2 + 4d(p_i, L)^2} \right\}$$

and now, let $y_i = \|x_i - x_{i+1}\|$ and let $z_i = d(p_i, L)$. Let $\ell = \sum_{i=1}^{n} y_i = \text{length}(L)$ and $d = \sum_{i=1}^{n} z_i$. The second term above is equivalent to

$$\frac{1}{\phi_1} \sum_{i=1}^{n} \sqrt{y_i^2 + 4z_i^2}$$
and so we consider the minimum of this over all $y_i$’s and $z_i$’s that sum to $\ell$ and $d$ respectively,

$$
\min_{y_i, z_i} \frac{1}{\phi_1} \sum_{i=1}^{n} \sqrt{y_i^2 + 4z_i^2} \quad \text{s.t.}
\sum_{i=1}^{n} y_i = \ell
\sum_{i=1}^{n} z_i = d
y_i, z_i \geq 0 \ \forall i,
$$

which is convex and separable. The solution lies at $y_i^* = \ell/n$ and $z_i^* = d/n$, which evaluates to $\sqrt{\ell^2 + 4d^2}$ as desired.

Problem (8) is NP-hard because the case where $\phi_1 \to \phi_0$ still corresponds to the Euclidean TSP, as did the original horsefly routing problem (1). This is unfortunate because, from the standpoint of approximation algorithms, one prefers a lower bound that is easy to compute (e.g., using a minimum spanning tree as a lower bound of a TSP tour). It is not obvious to the authors if such a lower bound for (1) exists.

As is standard in the continuous approximation paradigm, we now assume that customer demand points $p_i$ are independent samples from an absolutely continuous probability density function $f$ that is defined on $\mathcal{R}$. The summation in problem (8) then becomes an integral over $\mathcal{R}$, so that the continuous approximation to (8) takes the form

$$
\min_{L \in \text{Loop}(\mathcal{R})} \max \left\{ \frac{1}{\phi_0} \text{length}(L), \frac{1}{\phi_1} \sqrt{\text{length}(L)^2 + 4 \left( \int_{\mathcal{R}} f(x) d(x, L) dx \right)^2} \right\}. \quad (9)
$$

This problem is truly infinite-dimensional because its input is the density $f$ and its variable is a loop $L$; it is at least as difficult as its discrete counterpart (8), which as we have seen is already NP-hard. However, we can still characterize optimal trade-off between the two components in an asymptotic sense as follows:
Theorem 9. For fixed \( R, f, \phi_0, \) and \( \phi_1, \) the optimal objective value \( \text{OPT}(n) \) to problem (9) satisfies

\[
\text{OPT}(n) \sim \sqrt{\frac{n}{\phi_0 \phi_1}} \cdot \int_{R} \sqrt{f(x)} \, dx \cdot \Upsilon_1(\phi_0/\phi_1),
\]

as \( n \to \infty, \) where \( \Upsilon_1 : (0, 1) \to (1/\sqrt{2}, 1) \) is defined as

\[
\Upsilon_1(t) = \begin{cases} 
\frac{1}{\sqrt{2(1-t^2)^{1/4}}} & \text{if } t \leq 1/\sqrt{2} \\
\sqrt{t} & \text{otherwise.}
\end{cases}
\]

Proof. It is obvious that, if we multiply both \( \phi_0 \) and \( \phi_1 \) by a constant factor \( \alpha, \) then the solution to (9) does not change, and that \( \text{OPT}(n) \mapsto \frac{1}{\alpha} \text{OPT}(n). \) Thus, for notational convenience, we will scale by a factor of \( 1/\phi_1, \) which replaces \( \phi_1 \mapsto 1 \) and \( \phi_0 \mapsto t := \phi_0/\phi_1, \) with \( 0 < t < 1 \) by assumption. It is obvious that, as \( n \to \infty, \) the optimal solution to (9) must have length(\( L \)) \( \to \infty \) as well. Thus, by Theorem 4, we see that for optimal \( L \) whose length \( \ell \) is sufficiently large, we have

\[
\sqrt{\text{length}(L)}^2 + 4 \left( \int f(x) \, dx \right)^2 \sim \sqrt{\ell^2 + \frac{n^2}{4\ell^2} \left( \int \sqrt{f(x)} \, dx \right)^4},
\]

and therefore our (scaled) problem of interest is asymptotically equivalent to the problem

\[
\min_{\ell \geq 0} \max \left\{ \frac{1}{\ell}, \sqrt{\ell^2 + \frac{n^2}{4\ell^2} \left( \int \sqrt{f(x)} \, dx \right)^4} \right\}.
\]

This is a convex optimization problem in \( \ell, \) and its solution is given by

\[
\ell^* = \sqrt{n/2} \cdot \int_{R} \sqrt{f(x)} \, dx \cdot \min \left\{ 1, \frac{\sqrt{t}}{(1-t^2)^{1/4}} \right\}.
\]
and whose optimal objective value is

$$\max \left\{ \frac{1}{t} \ell, \sqrt{\ell^2 + \frac{n^2}{4t^2} \left( \int_{\mathcal{R}} \sqrt{f(x)} \, dx \right)^2} \right\}_{\ell = \ell^*} = \begin{cases} \frac{1}{\sqrt{2} (1-t^2)^{1/4}} & \text{if } t \leq 1/\sqrt{2} \\ \sqrt{t} & \text{otherwise} \end{cases}$$

The objective values from (10) are obtained precisely by re-scaling the problem back to its original form, i.e. dividing the above by $\phi_1$ and substituting $t = \phi_0/\phi_1$.

### 3.2 An upper bound

In the preceding section, we found a lower bound of the continuous approximation to the horsefly routing problem by bounding the cost of a horsefly tour whose truck uses a “zig-zag” route (or a “spiral” route, and so on); this is where the second term in (9) came from. In order to determine an upper bound, we will again have our truck use a “zig-zag” or “spiral” pattern, but we will (obviously) bound the time to completion from above. We first observe that one can obtain a crude upper bound for (1) by replacing the $\max\{\cdot, \cdot\}$ term with a sum, which gives

$$\min \sum_{x_1, \ldots, x_n, \sigma \in S_n} \frac{1}{\phi_0} \sum_{i=1}^n \| x_{\sigma(i)} - x_{\sigma(i+1)} \| + \frac{2}{\phi_1} \sum_{i=1}^n \| x_i - p_i \| ;$$

this is nothing more than the total time required to complete a horsefly route when one is subject to an additional constraint that the truck must remain stationary whenever the UAV is away from the vehicle. By replacing the finite point set $p_1, \ldots, p_n$ with a probability density as we did in the previous section, the continuous approximation equivalent of the above is given by

$$\min_{\mathcal{L} \in \text{Loop}(\mathcal{R})} \frac{1}{\phi_0} \text{length}(\mathcal{L}) + \frac{2n}{\phi_1} \int_{\mathcal{R}} f(x) d(x, \mathcal{L}) dx ,$$
and it is easy to see that by again using Theorem 4, this problem is asymptotically equivalent to

\[
\min_{\ell \geq 0} \frac{1}{\phi_0} \ell + \frac{n}{2\phi_1^2} \left( \int_{\mathcal{R}} \sqrt{f(x)} \, dx \right)^2,
\]

which gives an optimal solution of \( \sqrt{\frac{2n}{\phi_0 \phi_1}} \cdot \int_{\mathcal{R}} \sqrt{f(x)} \, dx \) obtained by solving for \( \ell^* = \sqrt{\frac{n \phi_0}{2 \phi_1}} \cdot \int_{\mathcal{R}} \sqrt{f(x)} \, dx \). This is comparable to our lower bound (10) from Theorem 9.

With a bit of extra work we can derive a tighter bound, using the same \( \ell^* \):

**Theorem 10.** Suppose that \( \mathcal{L} \) is a “zig-zag” or “spiral” tour of probability density \( f \) in a compact planar region \( \mathcal{R} \), having length \( \sqrt{\frac{n \phi_0}{2 \phi_1}} \cdot \int_{\mathcal{R}} \sqrt{f(x)} \, dx \). The expected time to completion of a horsefly tour that uses \( \mathcal{L} \) as a truck tour and visits \( n \) independent samples from \( f \) is asymptotically bounded above by

\[
\sqrt{\frac{n}{\phi_0 \phi_1}} \cdot \int_{\mathcal{R}} \sqrt{f(x)} \, dx \cdot \Upsilon_2(\phi_0/\phi_1) \tag{11}
\]

as \( n \to \infty \), where \( \Upsilon_2 \) is defined as

\[
\Upsilon_2(t) = \frac{1}{\sqrt{2}} \int_0^1 \int_0^1 \int_0^\infty (1/t) e^{-u/t} \max \left\{ u/t, \sqrt{u^2 + (v_1 + v_2)^2} \right\} \, du \, dv_1 \, dv_2
\]

and shown in Figure 5. The function \( \Upsilon_2(t) \) satisfies \( 0.99 < \Upsilon_2(t) < 1.11 \) for all \( t \in (0, 1) \).

**Proof.** To start, the bounds on \( \Upsilon_2 \) can be obtained by taking a piecewise linear approximation to the \( \sqrt{u^2 + (v_1 + v_2)^2} \) term in the integrand, which allows us to evaluate the integral analytically.
Figure 6: Figure (a) shows a zig-zag loop $\mathcal{L}$, the (uniform) points $p_i$, and their projections $p'_i$ onto $\mathcal{L}$. Figure (b) shows the points $x_i$ that minimize $\|p_{i-1} - x_i\| + \|x_i - p_i\|$ (only $x_1$, $x_2$, and $x_3$ are labelled, for purposes of brevity).

The particular bounds in question were obtained by taking piecewise linear upper and lower bounds of the term in question consisting of 32 components.

To derive the main result, it will suffice to consider the case where $f$ is uniform on $\mathcal{R}$, which has area 1, whence $\iint_{\mathcal{R}} \sqrt{f(x)} \, dx = 1$; this is because we can apply the same argument as in the proof of Theorem 4 where we approximated $f$ with a collection of piecewise constant step functions.

We will describe a particular scheme for determining the UAV’s trajectories with respect to the “zig-zag” loop $\mathcal{L}$ (or a “spiral”, or any other similar configuration), which we call Heron’s heuristic, named for the famous theorem that describes the shortest path between two points that touches a given line [28]. This scheme, which is shown in Figure 6, consists of the following steps:

1. Let $\mathcal{L}$ be a “zig-zag” or “spiral” loop through $\mathcal{R}$ whose length is $\ell := \sqrt{n\phi_0/(2\phi_1)}$ (for the non-uniform case, $\mathcal{L}$ should have length $\sqrt{n\phi_0/(2\phi_1)} \cdot \iint_{\mathcal{R}} \sqrt{f(x)} \, dx$).

2. For each of the $n$ samples $p_i$ drawn from $f$, let $p'_i$ be the nearest point on $\mathcal{L}$ to $p_i$. 

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Figure 7: Each of the two figures above shows a portion of $\mathcal{L}$ together with a pair of consecutive points $p_{i-1}, p_i$, as well as the projections $p'_{i-1}, p'_i$ and the launch site $x_i$ that minimizes $\|p_{i-1} - x_i\| + \|x_i - p_i\|$. In (a), we see that $x_i$ is merely the intersection of $\mathcal{L}$ with the line segment joining $p_{i-1}$ to $p_i$, because $p_{i-1}$ and $p_i$ are on opposite sides of $\mathcal{L}$. In (b), as proven in Heron’s shortest distance theorem [28], the point $x_i$ is obtained by reflecting $p_i$ about $\mathcal{L}$ and taking the same intersection as in (a).

3. Order the samples $p_i$ according to the sequence in which the points $p'_i$ appear on $\mathcal{L}$ (using an arbitrary starting point and orientation).

4. For each consecutive pair of samples $p_{i-1}, p_i$, let $x_i$ be the point on $\mathcal{L}$ situated between $p'_{i-1}$ and $p'_i$ that minimizes $\|p_{i-1} - x_i\| + \|x_i - p_i\|$. This is the “launch site” where the UAV departs from the truck and heads towards $p_i$.

Note that the points $p'_i$ become uniformly distributed along $\mathcal{L}$ as $n \to \infty$ (the only discrepancies arise because of the “kinks” in $\mathcal{L}$ that exist near the boundary of $\mathcal{R}$, whose contribution becomes negligible as $\mathcal{L}$ becomes long); more formally, we have convergence in distribution of these quantities. It is therefore a standard result of stochastic processes that the distance along $\mathcal{L}$ between arbitrarily chosen consecutive points $p'_{i-1}$ and $p'_i$ can be approximated by an exponential distribution, with rate parameter $\lambda = n/\ell = \sqrt{2\phi_1/\phi_0}$; see Section C of the online supplement for an ad hoc proof (and a precise phrasing) of this statement. Similarly, we also note that the distance between each sample $p_i$ and $\mathcal{L}$ converges to a uniform distribution between 0 and $1/(2\ell) = \sqrt{\phi_1/2n\phi_0}$ (again, the only discrepancies are due to the boundary of $\mathcal{R}$). Thus, as illustrated in Figure 7, the expected amount of service time required for each customer is the maximum of the amount of time that the truck drives and the amount of time that the UAV travels, which is given by
where $Z \sim \exp\left(\sqrt{2\phi_1 n/\phi_0}\right)$ and $Y_1, Y_2 \sim \text{uniform}(0, \sqrt{\frac{\phi_1}{2n\phi_0}})$ independently. Note that Figure 7 shows that, by virtue of similar triangles and Heron’s theorem, we do not have to concern ourselves with the location of the point $x_i$ where the truck and the UAV meet (in particular, the expectation above does not depend on such a value). The total time needed to complete the tour is obtained by multiplying the above expectation by $n$ and making the integral substitutions $u = z\sqrt{2n\phi_0/\phi_1}$ and $v_i = y_i\sqrt{2n\phi_0/\phi_1}$:

$$n\mathbb{E} \max \left\{ \frac{1}{\phi_0} Z, \frac{1}{\phi_1} \sqrt{Z^2 + (Y_1 + Y_2)^2} \right\} = \sqrt{\frac{n}{\phi_0\phi_1}} \cdot \Upsilon_2(\phi_0/\phi_1)$$

as desired.

3.3 Additional remarks and variations

By way of Theorems 9 and 10, we have now derived upper and lower bounds of a continuous relaxation of (1) that are proportional to $\sqrt{\frac{n}{\phi_0\phi_1}} \cdot \int\int_{\mathcal{R}} \sqrt{f(x)} \, dx$. Thus, we adopt the expression

$$\text{Service time with UAVs} \approx \beta' \sqrt{\frac{n}{\phi_0\phi_1}} \cdot \int\int_{\mathcal{R}} \sqrt{f(x)} \, dx$$

for some constant $\beta'$. By Theorem 2, we see that the amount of time needed to visit $n$ customers sampled from $f$ using only a truck (and no UAVs) is asymptotically equal to $\beta \sqrt{\frac{n}{\phi_0}} \cdot \int\int_{\mathcal{R}} \sqrt{f(x)} \, dx$, where $\beta$ is the TSP constant. Hence, the percent improvement that is gained by augmenting a
truck with a UAV can be approximated as

\[
\frac{\text{Service time without UAVs}}{\text{Service time with UAVs}} \approx \frac{\beta \sqrt{n}}{\sqrt{\phi_0} \phi_0} \cdot \int_{R} \frac{\sqrt{f(x)} \, dx}{\sqrt{f(x)} \, dx} = \alpha \sqrt{\frac{\phi_1}{\phi_0}},
\]

(12)

where \( \alpha = \beta/\beta' \). Thus, we hypothesize that the gains in efficiency due to introducing a UAV are proportional to \( \sqrt{\phi_1/\phi_0} \), and we will estimate \( \alpha \) in Section 4.1. There are also a number of variations of this problem that are worth considering:

3.3.1 Non-Euclidean distances

It is important to recognize that, while UAV trajectories are always measured in a Euclidean sense, the truck paths are not (since they are constrained by a road network). One way to compensate for a heterogeneous road network is by computing an “adjusted speed” \( \phi_0 \) for the truck as follows: initially, given the \( n \) locations of our customers, we compute a TSP tour of those locations (using the road network) and report the length \( \ell_0 \) of that tour (with respect to the road network) and the amount of time \( t_0 \) needed to visit those locations. An obvious estimate for \( \phi_0 \) would then be to take \( \ell_0/t_0 \). A better “adjusted” estimate is obtained by defining \( \ell'_0 \) to be the Euclidean length of the tour of the \( n \) points when visited in the same sequence as in the TSP tour on the road network, and estimating \( \phi_0 = \ell'_0/t_0 \). This results in smaller values of \( \phi_0 \) because \( \ell'_0 \leq \ell_0 \). In other words, we treat the truck as if it is travelling in a Euclidean sense, but at a sufficiently slow speed that the amount of time to travel from one point to the next is the same as if the truck had been using a road network.

3.3.2 Visiting clients with the truck

The preceding models assume that the truck serves exclusively as a mobile launching pad for the UAV and never visits destinations \( p_i \) itself. In practice, a situation may arise in which it is more efficient for the truck to visit certain destinations, and we can obtain suitable lower and upper
bounds without too much additional work.

**A lower bound** By weakening our original lower bound from Claim 8, we can obtain the following lower bound that applies when the truck is permitted to visit the points $p_i$:

*Claim 11.* The optimal objective value to the problem

\[
\begin{align*}
\min_{x_1, \ldots, x_n, \sigma \in S_n} \max & \left\{ \frac{1}{\phi_0} \sum_{i=1}^{n} \| x_{\sigma(i)} - x_{\sigma(i+1)} \|, \frac{2}{\phi_1} \sum_{i=1}^{n} \| x_i - p_i \| \right\} \\
\end{align*}
\]  

is a lower bound of the time to completion of a horsefly tour of $p_1, \ldots, p_n$, when the truck is permitted to provide service to some points $p_i$.

*Proof.* Let $L$ denote the optimal tour that the truck takes, which may include some points $p_i$. Certainly, the time to completion of the horsefly tour is at least $\frac{1}{\phi_0} \text{length}(L)$. For each point $p_i$, let $p_i'$ be the closest point on $L$ to $p_i$ (we have $p_i = p_i'$ for points visited by the truck), so that the total amount of time that the UAV is in flight is at least $\frac{2}{\phi_1} \sum_{i=1}^{n} \| p_i' - p_i \|$, which is therefore also a lower bound of the time to completion. The proof is complete by noting that, if we set $x_i = p_i'$ for all $i$ and let $\sigma$ be the order in which the points $p_i'$ appear on $L$, the objective value of (13) is precisely the same as the two lower bounds we have described in terms of $L$.

As suggested in the proof above, we can equivalently write problem (13) in terms of optimization over loops,

\[
\min_{L \in \text{Loop}(\mathcal{R})} \max \left\{ \frac{1}{\phi_0} \text{length}(L), \frac{2}{\phi_1} \sum_{i=1}^{n} d(p_i, L) \right\},
\]

and by replacing the summation with an integral we obtain the continuous approximation

\[
\min_{L \in \text{Loop}(\mathcal{R})} \left\{ \frac{1}{\phi_0} \text{length}(L), \frac{2n}{\phi_1} \int \int_{\mathcal{R}} f(x)d(x, L)dx \right\}.
\]
Applying Theorem 4, this problem is asymptotically equivalent to

\[
\min_{\ell \geq 0} \max \left\{ \frac{\ell}{\phi_0}, \frac{n}{2\phi_1\ell} \left( \iint_{\mathcal{R}} \sqrt{f(x)} \, dx \right)^2 \right\},
\]

which we readily observe has an optimal objective value of \( \sqrt{\frac{n}{2\phi_0\phi_1}} \cdot \iint_{\mathcal{R}} \sqrt{f(x)} \, dx \), realized by setting \( \ell^* = \sqrt{\frac{n}{2\phi_0\phi_1}} \cdot \iint_{\mathcal{R}} \sqrt{f(x)} \, dx \).

**An upper bound** Our upper bound is obtained in a very similar fashion to the derivation of Theorem 10:

**Claim 12.** Suppose that \( \mathcal{L} \) is a “zig-zag” or “spiral” tour of probability density \( f \) in a compact planar \( \mathcal{R} \), having length \( \sqrt{n\phi_0/(2\phi_1)} \cdot \iint_{\mathcal{R}} \sqrt{f(x)} \, dx \). Suppose that a truck and a UAV undertake a horsefly tour in which the truck traverses \( \mathcal{L} \), with the possibility of making detours to visit some of the samples from \( f \). The expected time to completion of this tour is asymptotically bounded above by

\[
\sqrt{\frac{n}{\phi_0\phi_1}} \cdot \iint_{\mathcal{R}} \sqrt{f(x)} \, dx \cdot \Upsilon_3(\phi_0/\phi_1)
\]
as \( n \to \infty \), where \( \Upsilon_3(t) \) is the function shown in Figure 8. The function \( \Upsilon_3(t) \) satisfies \( 0.91 < \Upsilon_3(t) < 0.97 \) for all \( t \in (0, 1) \).

**Proof.** As in the proof of Theorem 10, it will suffice to consider the case where \( f \) is uniform and \( \mathcal{R} \) has area 1. We saw in the proof of Theorem 10 that the expected amount of time to completion
for each sample in the original Heron heuristic was given by

\[
E_{\max} \left\{ \frac{1}{\phi_0} Z, \frac{1}{\phi_1} \sqrt{Z^2 + (Y_1 + Y_2)^2} \right\},
\]

where \( Z \sim \exp(\sqrt{2\phi_1 n/\phi_0}) \) represented the distance along \( L \) between consecutive points \( p'_{i-1}, p'_i \) and \( Y_1, Y_2 \sim \text{uniform}(0, \sqrt{\phi_1 / 2\phi_0}) \) independently. We will show how to make a tour that possibly uses “detours” in which the truck visits certain clients \( p_i \) by considering triplets of samples \( p_{i-1}, p_i, p_{i+1} \).

As explained in Section C.1 of the online supplement, we can approximate the distances from \( p'_{i-1} \) to \( p'_i \) and from \( p'_i \) to \( p'_{i+1} \) as a pair of independent exponential random variables \( Z_1, Z_2 \) with the same parameter \( \lambda = \sqrt{2\phi_1 n/\phi_0} \), and we now have three independent uniform random variables \( Y_1, Y_2, Y_3 \) between 0 and \( \sqrt{\phi_1 / 2\phi_0} \). By comparison with the diagram in Figure 7, there are now a multitude of scenarios to consider, two of which are shown in Figure 9. Whereas the costs associated with Figure 7 were easy to write out as an integral in three variables, the costs associated with Figure 9 are an extremely complicated five-dimensional integral, and so our function \( \Upsilon_3(t) \) was determined using numerical integration.

\( \square \)

### 3.3.3 UAV battery constraints

An important real-world limitation of horsefly systems is the battery life of a UAV, which can be expressed as a limit on the aggregate amount of time that the UAV is airborne; this means that there may be some time periods during which the truck drives while containing the UAV. It is easy
to modify problem (1) to take such a constraint into account; the only difference is that we must now distinguish between “launch sites” $x_i$ and “landing sites” $y_i$:

$$\minimize_{x_1, \ldots, x_n, y_1, \ldots, y_n, \sigma \in \mathcal{S}_n} \sum_{i=1}^n \max \left\{ \frac{1}{\phi_0} \|x_{\sigma(i)} - y_{\sigma(i)}\|, \frac{1}{\phi_1} \left( \|x_{\sigma(i)} - p_\sigma(i)\| + \|p_\sigma(i) - y_{\sigma(i)}\| \right) \right\}$$

$$+ \sum_{i=1}^n \frac{1}{\phi_0} \|y_{\sigma(i)} - x_{\sigma(i+1)}\|$$

s.t.

$$\sum_{i=1}^n \frac{1}{\phi_1} \left( \|x_{\sigma(i)} - p_\sigma(i)\| + \|p_\sigma(i) - y_{\sigma(i)}\| \right) \leq \tau,$$

where $\tau$ is the battery life of the UAV. Note that, for fixed $\sigma$, the above is a convex optimization problem in the $x_i$’s and $y_i$’s (a second order cone program). Figures 10 and 11 show solutions to this problem.
The behavior of problem (14) for small values of $\tau$ is particularly interesting because most UAVs that are presently on the market have a very short battery life relative to a truck. It turns out that we can actually describe the optimal solution to (14) exactly in the limit as $\tau \to 0$ under very general conditions (without appealing to any continuous approximations whatsoever):

**Theorem 13.** Suppose that the TSP tour of points $p_1, \ldots, p_n$ contains an angle $\theta$ that satisfies $\theta \leq 2 \arcsin(\phi_0/\phi_1)$. Then for sufficiently small $\tau$, we have

$$\text{OPT} = \frac{1}{\phi_0} \text{TSP} - (\phi_1/\phi_0 - 1)\tau,$$

where $\text{OPT}$ is the duration of an optimal (capacitated) horsefly tour and $\text{TSP}$ is the length of the TSP tour of the $p_i$'s.

**Proof.** See Section D of the online supplement. □

For general values of $\tau$, battery life constraints act as a continuum between a pure TSP tour of the destinations using the truck and an unrestricted horsefly tour. This is because the TSP simply corresponds to setting $\tau = 0$, whose duration is $\beta \sqrt{\frac{n}{\phi_0}} \cdot \int R \sqrt{f(x)} \, dx$, whereas the original horsefly tour corresponds to setting $\tau \to \infty$ (or at least, selecting a large value of $\tau$), whose duration is $\beta' \sqrt{\frac{n}{\phi_0\phi_1}} \cdot \int R \sqrt{f(x)} \, dx$. It is useful to re-scale the battery life constraint in terms of the duration of the unrestricted horsefly tour:

$$\tau = \rho \beta' \sqrt{\frac{n}{\phi_0\phi_1}} \cdot \int R \sqrt{f(x)} \, dx,$$

with $0 \leq \rho \leq 1$ (see Section 4.2 of [20] or the paper [21], which do the same kind of re-scaling for the capacitated vehicle routing problem). Applying this re-scaling, we approximate the duration of a capacitated horsefly tour as $\Gamma(\phi_0, \phi_1, \rho) \cdot \sqrt{n} \int R \sqrt{f(x)} \, dx$, for some function $\Gamma(\cdot, \cdot, \cdot)$ such that $\Gamma(\phi_0, \phi_1, 0) = \beta/\phi_0$ and $\Gamma(\phi_0, \phi_1, 1) = \beta'/\sqrt{\phi_0\phi_1}$. By a standard scaling argument, we have
\[ \Gamma(\phi_0, \phi_1, \rho) = \frac{1}{\phi_1} \Gamma(\phi_0/\phi_1, 1, \rho), \] and so we can approximate the duration as simply \( \Lambda(\phi_0/\phi_1, \rho) \cdot \frac{\sqrt{n}}{\phi_1} \int_R \sqrt{f(x)} \, dx \) for some function \( \Lambda(\cdot, \cdot) \). As an aside, Theorem 13 allows us to describe the partial derivative \( \frac{\partial}{\partial \rho} \Lambda(t, \rho) \) evaluated at \( \rho = 0 \). We therefore adopt the expression

\[
\frac{\text{Service time without UAVs}}{\text{Service time with constrained UAVs}} \approx \frac{\beta \frac{\sqrt{n}}{\phi_0} \cdot \int_R \sqrt{f(x)} \, dx}{\Lambda(\phi_0/\phi_1, \rho) \cdot \frac{\sqrt{n}}{\phi_1} \int_R \sqrt{f(x)} \, dx} = \beta \frac{\phi_1}{\phi_0} \Lambda(\phi_0/\phi_1, \rho) \tag{15}
\]

for some function \( \xi_{\text{batt}}(\cdot, \cdot) \), which we will estimate in Section 4.1.

## 4 Computational results

In this section we conduct two computational experiments. The first experiment is done in the unit square with uniformly distributed demand, which we use to estimate the coefficient \( \alpha \) introduced in equation (12) and the function \( \xi_{\text{batt}}(\cdot, \cdot) \) introduced in (15). The second experiment uses these estimates of \( \alpha \) to make predictions about the improvements in efficiency when demand follows a non-uniform distribution and real-time driving information on a road network is used.

### 4.1 Experiments using Euclidean distances

In our first experiment, we sample \( n = 500 \) points uniformly in the unit square, we fix a truck speed of \( \phi_0 = 1 \), and we allow the UAV speed \( \phi_1 \) to vary with \( \phi_1 \in \{1.5, 2, 2.5, 3, 5\} \). Because of the difficulty of solving a given problem instance to optimality, we use a simple heuristic for solving problem (1): first, we set the permutation \( \sigma \) to be the same as the TSP tour of the points \( p_i \), and we solve the convex optimization problem over the \( \lambda_i \)'s that results from fixing \( \sigma \). We then perform 2-OPT and 3-OPT exchanges (i.e. exchanging pairs and triplets in \( \sigma \)) until no additional improvements can be made; note that, for every such exchange, we must solve a new convex optimization problem over the \( x_i \)'s, so this procedure can take a very long time to complete;
for this reason, we terminated our search after performing 20000 exchanges, each of which was
determined by selecting elements of $\sigma$ uniformly at random. Table 1a shows the estimates of $\alpha$ for
these 5 values of $\phi_1$ (over a total of 200 experiments for each value of $\phi_1$), which tend to stay close
to 1, with a slight downward trend as $\phi_1$ increases. Each estimate $\hat{\alpha}$ is obtained by computing the
TSP tour of the sampled points $p_1, \ldots, p_{500}$ as well as the horsefly tour that results from performing
the heuristic above, and then setting

$$
\hat{\alpha} = \frac{\text{Service time without UAVs}}{\text{Service time with UAVs} \cdot \sqrt{\phi_1/\phi_0}}.
$$

(16)

In addition, as in Section 3.3.2, we also estimate the improvement in efficiency when UAVs are
used and the truck is permitted to visit destination points. Since Section 3.3.2 establishes that the
proportionality to $\sqrt{\phi_1/\phi_0}$ is still present, we define a ratio $\alpha'$ in entirely the same way as in the
previous paragraph:

$$
\hat{\alpha}' = \frac{\text{Service time without UAVs}}{(\text{Service time with UAVs + truck visits}) \cdot \sqrt{\phi_1/\phi_0}}.
$$

The problem of finding an optimal horsefly tour in this setting is particularly difficult; whereas
problem (1) reduces to a convex optimization problem when the sequence $\sigma$ is fixed, we have no
such luxury here. Therefore, we simply apply a greedy insertion rule to the procedure from the
previous paragraph: given a horsefly tour, we iterate through each point $p_i$ that the UAV visits and
ask if the tour can be shortened by assigning $p_i$ to the truck. Then, the point $p_i$ whose re-assignment
gives the maximum improvement is assigned to the truck, and the process repeats. Table 1b shows
the estimates of $\alpha'$ for the same 5 values of $\phi_1$, again over 200 experiments for each value of $\phi_1$.
Note that $\alpha'$ shows a significant dependency on $\phi_1$ and varies depending on whether $\phi_1$ is close to
$\phi_0$. This is not surprising because the truck will almost never visit points $p_i$ when $\phi_1$ is large, and
thus we expect $\alpha'$ to be very close to $\alpha$ for large values of $\phi_1$. 

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Finally, to test the impact of limited battery life as in Section 3.3.3, we use the same values of $\phi_0$, $\phi_1$, and $n$, as well as using $\rho \in \{0.1, 0.2, \ldots, 0.9\}$, and we set

$$\hat{\xi}_{batt}(\phi_0/\phi_1, \rho) = \frac{\text{Service time without UAVs} \cdot (\phi_0/\phi_1)}{\text{Service time with constrained UAVs}}$$  (17)

the average values of $\hat{\xi}_{batt}(\cdot, \cdot)$ are shown in Table 1c.

### 4.2 Experiments using road network distances

In our second experiment, we use the results of the preceding section to predict the changes in service time when UAVs are introduced to a truck that visits a collection of $n$ destination points in a map of Pasadena, California, with $n \in \{25, 50, 100\}$. These $n$ destination points are sampled uniformly from the centers of all 1734 US census blocks that belong to Pasadena, as shown in
Figure 12: Figure (a) shows the TSP tour (with respect to the road network) of \( n = 25 \) points sampled from 1734 US census blocks located in Pasadena, California. Figure (b) shows the horsefly tour computed using the two-stage heuristic from Section 4.1.

Figure 12, which were obtained from [1]. Next, we compute a TSP tour of those \( n \) points with respect to the road network distance, as shown in Figure 12a; the length and duration of this TSP tour is estimated using the Google Maps Directions API [3], the Google Distance Matrix API [23], and the Concorde TSP Solver [2]. Finally, we apply the same 2- and 3-OPT heuristic from the previous section to compute a horsefly tour of those same points, as shown in Figure 12b. The “adjusted” truck speed \( \phi_0 \) is generally around \( 20 \text{km/hr} \) in these experiments, and the UAV speed \( \phi_1 \) satisfies \( \phi_1 \in \{30 \text{km/hr}, 40 \text{km/hr}, 50 \text{km/hr}, 60 \text{km/hr} \} \).

Based on our preceding analysis (namely, equation (16)), we expect that the two service times should satisfy

\[
\text{Service time with UAVs} = \frac{\text{Service time without UAVs}}{\alpha \cdot \sqrt{\phi_1/\phi_0}},
\]

and we will use the values of \( \alpha \) from the previous section in Table 1a. Table 2a shows the relative prediction error for 10 experiments for each pair of \( n \in \{25, 50, 100\} \) and \( \phi_1 \in \{30 \text{km/hr}, 40 \text{km/hr}, 50 \text{km/hr}, 60 \text{km/hr} \} \), i.e. the “predicted” service times divided by the actual service times; the complete raw data can be found in Table 3 of the Online Supplement. These confirm that our approximation is indeed a sensible “back-of-the-envelope” estimate of the true service time of a horsefly tour. Somewhat surprisingly, the overall quality of prediction does not appear to depend on \( n \). In one sense, this is
surprising because it is generally well-known that continuous approximation schemes can perform poorly with low sample sizes (see for example Table 16.7 of [7]). However, it is likely that this is being mitigated at present because we do not directly use a square root-type approximation to make these predictions; rather, the predictions are based on the actual duration of a TSP tour of the points, not a square-root approximation thereof.

Table 2b shows the results for the same experiment as the preceding paragraph, for the case where the truck can also visit the points $p_i$. Here the predictions are made using equation (18), substituting $\alpha'$ as derived in the previous section. The predictions are accurate for high values of $\phi_1$, but are consistently too low (i.e., too optimistic) for low values of $\phi_1$. This is likely due to the fact that the truck is only likely to visit a destination $p_i$ if it is very close to some other destination $p_j$, and the distortion between Euclidean distances versus road network distances can be particularly severe in those cases. On the other hand, for all values of $\phi_1$, the percent error does not depend on $n$; this is encouraging because it still suggests that the overall proportionality is the correct one. The complete raw data can be found in Table 4 of the Online Supplement.

Tables 2c through 2e, and Tables 5 through 7 of the Online Supplement, show the prediction error that results when we impose a battery life constraint of a half hour, an hour, or two hours. In order to predict the duration of a horsefly tour, we adapt equation (17) to obtain

$$\text{Service time with constrained UAVs} = \frac{\text{Service time without UAVs} \cdot (\phi_0/\phi_1)}{\xi_{\text{batt}}(\phi_0/\phi_1, \rho)}.$$  

Note, however, that the values of $\xi_{\text{batt}}(\cdot, \cdot)$ are defined in terms of $\rho$ (which is a relative duration defined by a comparison to the duration of an unconstrained horsefly tour), whereas our battery life constraints are given in absolute durations. Thus, we must also estimate $\rho$, and we do so with the estimate

$$\hat{\rho} = \frac{\tau}{\text{Predicted service time with unconstrained UAVs}} = \frac{\hat{\alpha} \tau \cdot \sqrt{\phi_1/\phi_0}}{\text{Service time without UAVs}}.$$
by applying our previous prediction (18), with the estimated values $\hat{\alpha}$. These predictions are still good ballpark estimates, although they have a tendency to underestimate the true service time, especially for small values of $n$. This suggests that the predictor is overly optimistic in the sense that it expects the UAV to be used in a more efficient manner than the underlying road network truly permits; on the other hand, the low overall prediction error over a reasonably wide range of $n$ is encouraging.

5 Conclusions

By using an asymptotic theoretical analysis in the Euclidean plane as well as a collection of computational experiments, we have concluded that the improvement in efficiency due to augmenting a delivery truck with a UAV is related to the square root of the ratio of the speeds of the truck and the UAV. One of the weak points in our analysis is the fact that we use heuristic methods to compute the coordinated routes between the truck and the UAV, rather than a true globally optimal solution. While we are unaware at present of any techniques that find these solutions for the problem scales discussed in this paper, we expect them to become available in the coming years as interest in UAVs in logistics increases; a promising direction would be the use of tool such as guillotine subdivisions, which have previously been used to solve geometric instances of the TSP, Steiner tree, and $k$-minimum spanning tree problems [43].

References


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(a) Relative prediction errors for 120 road network experiments.

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(b) Relative prediction errors for 120 road network experiments in which the truck is also permitted to visit destination points.

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(c) Relative prediction errors for 120 road network experiments in which the drone has a battery life of a half hour, i.e. τ = 0.5.

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(d) Relative prediction errors for 120 road network experiments in which the drone has a battery life of one hour, i.e. τ = 1.

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<td>0.97</td>
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(e) Relative prediction errors for 120 road network experiments in which the drone has a battery life of two hours, i.e. τ = 2.

Table 2: Relative prediction errors for experiments conducted on the Pasadena road network.


Online supplement to “Coordinated logistics with a truck and a drone”

A Proof of Lemma 3

The assumption that $\mathcal{L}$ is polygonal is a valid one because the standard definition of the length of $\mathcal{L}$ is simply the limit of a discretization; if we say that $\mathcal{L}$ is the image of the continuous mapping $\gamma : [0, 1] \to \mathbb{R}^2$, then the length of $\mathcal{L}$ is defined as

$$\text{length}(\mathcal{L}) = \sup_{0=t_0<t_1<\cdots<t_M=1} \sum_{i=1}^{M-1} \|\gamma(t_i) - \gamma(t_{i-1})\|$$

where the supremum is taken over all possible partitions of $[0, 1]$ and $M$ is unbounded.

The assumption that $\mathcal{L}$ forms the boundary of a convex region is also straightforward: at the very least, we can certainly assume that $\mathcal{L}$ is simple, that is, that $\mathcal{L}$ does not intersect itself (since one can always “un-cross” a pair of intersecting edges of $\mathcal{L}$ in an obvious way). If $\mathcal{L}$ is not the boundary of a convex region, then (since we have assumed that $\mathcal{L}$ is polygonal) there must exist a pair of vertices $v_1, v_2$ of $\mathcal{L}$ such that $v_1$ and $v_2$ are adjacent on the convex hull $\text{Conv}(\mathcal{L})$ of $\mathcal{L}$, but not adjacent on $\mathcal{L}$ itself; see Figure 13a. If we let $s$ denote the component of $\mathcal{L}$ that lies between $v_1$ and $v_2$, then it is obvious that we can reflect $s$ about the line joining $v_1$ and $v_2$, to obtain a new curve $\mathcal{L}'$ with the same length as $\mathcal{L}$, as shown in Figure 13b. It is then an entirely straightforward argument to verify that, for any $\epsilon$, we have $\text{Area}(N_\epsilon(\mathcal{L}')) \geq \text{Area}(N_\epsilon(\mathcal{L}))$, which completes the proof.
Figure 13: Reflecting the polygonal chain \( s \) about the line joining \( v_1 \) and \( v_2 \).

### B Zig-zagging and spiralling

Here we give a precise description of the construction of a “zig-zag” or “spiral” tour for sufficiently large \( \ell \), where we assume that \( \mathcal{R} \) is convex with area \( A \). We describe two schemes in which one is given a length \( \ell \) and gives a loop \( \mathcal{L} \) such that length(\( \mathcal{L} \)) \( \leq \ell \) and such that \( \int_{\mathcal{R}} d(x, \mathcal{L}) \, dx \sim A^2/(4\ell) \) as \( \ell \to \infty \).

**Zig-zagging** We can represent \( \mathcal{R} \) as the region bounded by a convex function and a concave function of one variable, i.e. \( \mathcal{R} = \{ x = (x_1, x_2) \in \mathbb{R}^2 : f(x_1) \leq x_2 \leq g(x_1) \} \) for some convex \( f \) and concave \( g \), whence Area(\( \mathcal{R} \)) = \( A = \int_a^b g(t) - f(t) \, dt \) for appropriate horizontal bounds \( a \) and \( b \). Assume without loss of generality that \( a = 0 \) and \( b = 1 \), so that Riemann integration yields

\[
A = \int_0^1 g(t) - f(t) \, dt = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N g(i/N) - f(i/N) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \ell_i,
\]

where we define \( \ell_i := g(i/N) - f(i/N) \). Let \( s_i \) denote the line segment that corresponds to \( \ell_i \), and note that (provided that \( N \) is even) we can join the \( s_i \)'s together to form a loop \( \mathcal{L} \) whose length is at most \( \sum_{i=1}^N \ell_i + 2P \), where \( P \) is the perimeter of \( \mathcal{R} \) (a constant); Figure 14 shows how this is done. As \( N \to \infty \), the above can be equivalently written as \( \sum_{i=1}^N \ell_i \sim AN \).

Further note that the distance between a uniformly selected point \( x \in \mathcal{R} \) and its nearest segment \( s_i \) converges to a uniform distribution between 0 and \( 1/(2N) \), which establishes that
the expected distance between uniform $x$ and its nearest $s_i$ converges to $1/(4N)$, whence \[ \iint_{\mathcal{R}} d(x, \mathcal{L}) \, dx \sim A/(4N) \] as $N \to \infty$. Therefore, given $\ell$, we can set $N$ to be the largest even integer such that $\sum_{i=1}^{N} \ell_i + 2P \leq \ell$, and then form $\mathcal{L}$ by joining the appropriate segments $s_i$. We see that as $\ell \to \infty$, we have $N \to \infty$ as well, whence

$$\ell \sim \sum_{i=1}^{N} \ell_i \sim AN \implies \iint_{\mathcal{R}} d(x, \mathcal{L}) \, dx \sim \frac{A}{4N} \sim \frac{A}{4\ell} = \frac{A^2}{4\ell}$$

as desired.

**Spiralling** It will suffice to build a family of concentric loops $\mathcal{L} = \{\mathcal{L}_i\}$ within $\mathcal{R}$ such that the desired conditions hold, i.e. \[ \iint_{\mathcal{R}} \min_i d(x, \mathcal{L}_i) \, dx \sim A^2/(4\ell). \] This is sufficient because we can stitch these loops to form a single large loop $\mathcal{L}$ by taking convex combinations of consecutive loops, as suggested in Figure 15. This process does not increase the total length by more than the diameter $D$ of $\mathcal{R}$, that is, \[ \text{length}(\mathcal{L}) \leq \sum_i \text{length}(\mathcal{L}_i) + D. \] This construction is similar to the proof of Lemma 3. For any $\epsilon$, let $\mathcal{L}_\epsilon$ be the loop consisting of all points $x \in \mathcal{R}$ that are a distance of exactly $\epsilon$ away from the boundary of $\mathcal{R}$ (which would itself therefore be written...
as $\mathcal{L}_0$). The coarea formula [38] says that the area $A$ of $\mathcal{R}$ can be written as

$$A = \int_0^{\bar{\epsilon}} \text{length}(\mathcal{L}_\epsilon) \, d\epsilon$$

where $\bar{\epsilon}$ is the “radius” of $\mathcal{R}$, i.e. $\bar{\epsilon} = \max_{x \in \mathcal{R}} d(x, \mathcal{L}_0)$. Assume without loss of generality that $\bar{\epsilon} = 1$, so that Riemann integration yields

$$A = \int_0^{1} \text{length}(\mathcal{L}_\epsilon) \, d\epsilon = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \text{length}(\mathcal{L}_{i/N}) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ell_i,$$

where we define $\ell_i := \text{length}(\mathcal{L}_{i/N})$. As $N \to \infty$, the above can equivalently be written as $\sum_{i=1}^{N} \ell_i \sim AN$. Further note that the distance between a uniformly selected point $x \in \mathcal{R}$ and its nearest loop $\mathcal{L}_i$ converges to a uniform distribution between 0 and $1/(2N)$, which establishes that the expected distance between uniform $x$ and its nearest $\mathcal{L}_i$ converges to $1/(4N)$, whence $\iint_{\mathcal{R}} \min_x d(x, \mathcal{L}_i) \, dx \sim A/(4N)$ as $N \to \infty$. Therefore, given $\ell$, we can set $N$ to be the largest even integer such that $\sum_{i=1}^{N} \ell_i + D \leq \ell$, and then form $\mathcal{L}$ by joining consecutive concentric loops $\mathcal{L}_i$. We see that as $\ell \to \infty$, we have $N \to \infty$ as well, whence

$$\ell \sim \sum_{i=1}^{N} \ell_i \sim AN \implies \iint_{\mathcal{R}} d(x, \mathcal{L}) \, dx \sim \frac{A}{4N} \sim \frac{A}{4\ell} = \frac{A^2}{4\ell}$$

as desired.

C Proof of Theorem 10

Suppose that $\mathcal{L}$ is a loop with length $\ell := c\sqrt{n}$ (where we would have $c = \sqrt{\phi_0/(2\phi_1)}$ in Theorem 10) and suppose that points $Q_1, \ldots, Q_n$ are uniformly distributed (with respect to the arc length measure) on $\mathcal{L}$. If we select a point $Q_i$ arbitrarily, we see that the clockwise distance from $Q_i$ to any other point $Q_j$ is uniformly distributed between 0 and $\ell$. Therefore, the distance $D$ from $Q_i$
Figure 15: Joining a family of concentric loops (a) into a single loop (b).

to its nearest neighbor in the clockwise direction has a cdf $F(d)$ given by

$$F(d) = 1 - \left( \frac{\ell - d}{\ell} \right)^{n-1}.$$ 

We want to show that $D$ can be approximated by an exponential distribution with rate parameter $\lambda = \sqrt{n}/c$; more precisely, we will show that the random variable $D\sqrt{n}$ converges in distribution to an exponential distribution with rate parameter $1/c$:

$$\lim_{n \to \infty} \Pr(D\sqrt{n} \leq d) = \lim_{n \to \infty} \Pr(D \leq d/\sqrt{n}) = \lim_{n \to \infty} F(d/\sqrt{n}) = \lim_{n \to \infty} 1 - \left( \frac{c\sqrt{n} - d/\sqrt{n}}{c\sqrt{n}} \right)^{n-1} = \lim_{n \to \infty} 1 - \left( \frac{cn - d}{cn} \right)^{n-1} = 1 - e^{-d/c}$$

as desired.

C.1 A joint distribution

We can use an entirely analogous argument to establish that the distances between two consecutive points along $L$ can be approximated with a pair of independent exponential random variables. Again, as in the preceding argument, we know that the distance $D_1$ from an arbitrarily selected
point $Q_i$ to its nearest neighbor $Q_j$ in the clockwise direction has a cdf $F_1(d_1)$ given by

$$F_1(d_1) = 1 - \left( \frac{\ell - d_1}{\ell} \right)^{n-1},$$

and it is also easy to see that the distance $D_2$ from $Q_j$ to its nearest neighbor in the clockwise direction, conditioned on the event that $D_1 = d_1$, has a cdf $F_2(d_2|D_1 = d_1)$ given by

$$F_2(d_2|D_1 = d_1) = 1 - \left( \frac{\ell - d_1 - d_2}{\ell - d_1} \right)^{n-2}.$$  

By differentiating, we see that the corresponding probability density functions are given by

$$f_1(d_1) = \frac{n-1}{\ell^{n-1}} \cdot (\ell - d_1)^{n-2}$$

$$f_2(d_2|D_1 = d_1) = \frac{n-2}{(\ell - d_1)^{n-2}} \cdot (\ell - d_1 - d_2)^{n-3}$$

which therefore gives a joint cdf $F(d_1, d_2)$ given by

$$F(d_1, d_2) = \int_0^{d_1} \int_0^{d_2} f_1(d_1') f_2(d_2'|D_1 = d_1') \, dd_2 \, dd_1'$$

$$= \frac{(n-1)(n-2)}{\ell^{n-1}} \int_0^{d_1} \int_0^{d_2} (\ell - d_1 - d_2)^{n-3} \, dd_2 \, dd_1'$$

$$= 1 + \left( \frac{\ell - d_1 - d_2}{\ell} \right)^{n-1} - \left( \frac{\ell - d_1}{\ell} \right)^{n-1} - \left( \frac{\ell - d_2}{\ell} \right)^{n-1}$$

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We claim that the joint distribution on $D_1\sqrt{n}$ and $D_2\sqrt{n}$ converges to a joint cdf on two independent exponential random variables with rate parameter $1/c$:

$$\lim_{n\to\infty} \Pr(D_1\sqrt{n} \leq d_1 \cap D_2\sqrt{n} \leq d_2)$$
$$= \lim_{n\to\infty} \Pr(D_1 \leq d_1/\sqrt{n} \cap D_2 \leq d_2/\sqrt{n})$$
$$= \lim_{n\to\infty} F(d_1/\sqrt{n}, d_2/\sqrt{n})$$
$$= \lim_{n\to\infty} 1 + \left(\frac{cn - d_1}{cn - d_2}\right)^{n-1} - \left(\frac{cn}{cn - d_1}ight)^{n-1} - \left(\frac{cn}{cn - d_2}\right)^{n-1}$$
$$= 1 + e^{-(d_1 + d_2)/c} - e^{-d_1/c} - e^{-d_2/c}$$
$$= \left(1 - e^{-d_1/c}\right) \left(1 - e^{-d_2/c}\right)$$

as desired.

\section*{D Proof of Theorem 13}

Assume (purely for notational purposes) that the optimal permutation $\sigma^*$ is the identity and consider the sequence of stops traversed by the UAV:

$$x_1^* \rightarrow p_1 \rightarrow y_1^* \rightarrow x_2^* \rightarrow p_2 \rightarrow y_2^* \rightarrow \cdots \rightarrow x_n^* \rightarrow p_n \rightarrow y_n^* \rightarrow x_1^*.$$

Let $\ell_0 = \sum_{i=1}^n \|y_i^* - x_{i+1}^*\|$ and $\ell_1 = \sum_{i=1}^n \|x_i^* - p_i\| + \|p_i - y_i^*\|$, so that $\ell_0$ represents the length that is traversed jointly with the truck and the UAV and $\ell_1$ represents the length that is traversed exclusively by the UAV. Our battery life constraint says that $\ell_1 \leq \phi_1 \tau$. Certainly, we have $\ell_0 + \ell_1 \geq$
TSP, and therefore the optimal duration $\text{OPT}$ is bounded below by the linear program

$$\begin{align*}
\text{minimize} \quad & \frac{1}{\phi_0} \ell_0 + \frac{1}{\phi_1} \ell_1 \\
\text{s.t.} \quad & \ell_0 + \ell_1 \geq \text{TSP} \\
& \ell_1 \leq \phi_1 \tau .
\end{align*}$$

Using the fact that $\phi_0 < \phi_1$, standard arguments guarantee that $\ell_0^* = \text{TSP} - \phi_1 \tau$ and $\ell_1^* = \phi_1 \tau$, which tells us that $\text{OPT} \geq \frac{1}{\phi_0} \text{TSP} - (\phi_1/\phi_0 - 1)\tau$ for all $\tau$. To prove that equality holds when $\tau$ is small, we simply have the truck traverse the TSP tour of the $p_i$’s, and then “cut corners” with the UAV when the TSP tour makes an angle that is sufficiently acute (the condition that there exists $\theta$ such that $\theta \leq 2 \arcsin(\phi_0/\phi_1)$ merely asserts that there exists one such angle).

E Raw data from Section 4.2
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Table 3: The above table is divided into \( 3 \times 4 \) blocks, each corresponding to a choice of \( n \) and \( \phi_1 \). Each block contains 10 rows, each corresponding to a single experiment, and 4 columns labelled “T”, “P”, “H”, and “P/H”, which are interpreted as follows: column “T” is the duration, in hours, of a TSP tour of the \( n \) points when using road network distances and times as determined by the Google Distance Matrix API [23] and the Concorde TSP Solver [2]. Column “P” is the predicted duration of a horsefly tour of the \( n \) points, as predicted from equation (18). Column “H” is the actual duration of a horsefly tour of the \( n \) points, again using the the Google Maps Directions API [3], the Google Distance Matrix API [23], and the 2- and 3-OPT heuristic described in the previous section. The percent error of the prediction is given in the fourth column, which is labelled “P/H”, and shows the ratio of the predicted horsefly tour length to the actual horsefly tour length.
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Table 4: Results of computational experiments where the truck is permitted to visit destination points. All data fields are the same as in Table 3.
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<td>4.15 4.09 4.23 0.97</td>
<td>4.60 4.37 4.19 1.04</td>
</tr>
<tr>
<td></td>
<td>4.11 4.69 4.05 1.16</td>
<td>4.28 4.41 4.68 0.94</td>
<td>4.06 4.00 4.06 0.99</td>
<td>4.57 4.34 4.29 1.01</td>
</tr>
<tr>
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<td>4.24 4.84 4.83 1.00</td>
<td>4.58 4.73 4.82 0.98</td>
<td>3.96 3.90 3.93 0.99</td>
<td>4.28 4.07 4.09 1.00</td>
</tr>
<tr>
<td></td>
<td>4.05 4.44 4.70 0.95</td>
<td>4.22 4.36 4.46 0.98</td>
<td>4.09 4.03 4.04 1.00</td>
<td>4.02 3.82 3.91 0.98</td>
</tr>
<tr>
<td></td>
<td>4.72 5.39 5.24 1.03</td>
<td>4.40 4.54 4.54 1.00</td>
<td>4.28 4.22 4.26 0.99</td>
<td>5.13 4.88 4.14 1.18</td>
</tr>
</tbody>
</table>

Table 5: Results of computational experiments when UAV battery life is limited to a half hour, i.e. \( \tau = 0.5 \). All data fields are the same as in Table 3.
Table 6: Results of computational experiments when UAV battery life is limited to an hour, i.e. $\tau = 1$. All data fields are the same as in Table 3.
Table 7: Results of computational experiments when UAV battery life is limited to two hours, i.e. \( \tau = 2 \). All data fields are the same as in Table 3.